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THE EQUATIONS OF MOTION OF A TUMBLING RE-ENTRY BODY

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ABSTRACT

~~This Report presents~~ a set of Equations, which may be used to describe the re-entry trajectory of a coasting tumbling rocket. The Equations of motion are kept as general as possible, with the e ptions that the azimuth coordinate is linearized, an ; and Magnus forces are neglected.

are presented

I. COORDINATE SYSTEMS

Fixed at the center of the earth is the origin of an XYZ coordinate system (Fig. 1). The Y axis passes through the point at which the initial conditions are established, the XY plane contains the target, and the Z axis completes a right-hand coordinate system.

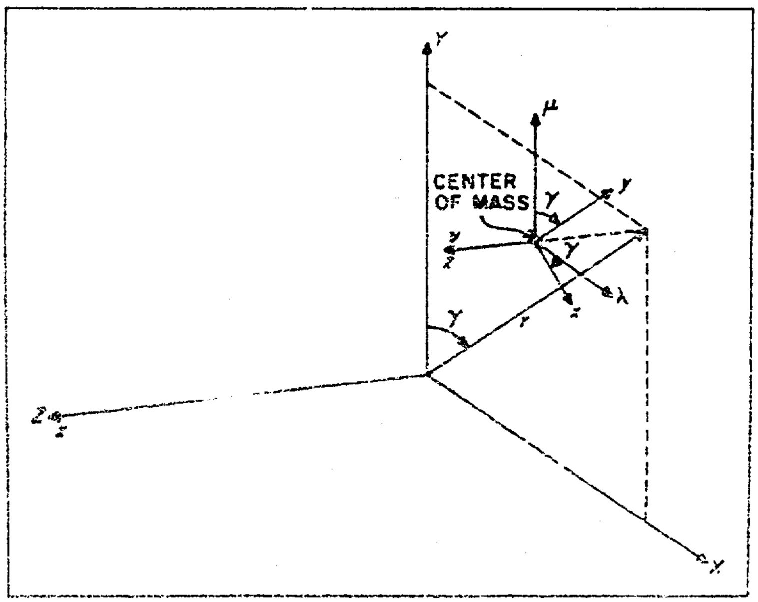

Fig. 1. Space Coordinate System

The rocket center of mass is located by the cylindrical coordinates r, γ, z: r is the distance from the origin to the projection of the rocket center of mass on the XY plane, γ is the angle between Y and r, and z is identical to Z. There are two sets of space axes located with their origins at the rocket center of mass. The $\lambda\mu\nu$ coordinate system is so oriented that the axes are parallel, respectively, to the XYZ axes. The xyz coordinate system is so oriented that the xy plane is parallel to the XY plane, and y is parallel to r. Also located with its origin at the rocket center of mass is a set of $\xi\eta\zeta$ body axes (Fig. 2). The roll axis is ξ, the yaw axis is η, and the pitch axis is ζ.

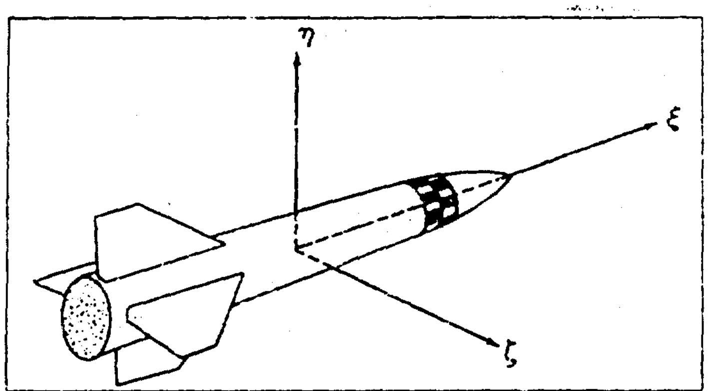

Fig. 2. Body Coordinate System

The orientation of the body with respect to the $\lambda\mu\nu$ coordinate system is described by the Euler angles ϕ, θ, ψ, in the following manner (Fig. 3)[1]. Starting with $\xi\eta\zeta$ axes respectively coincident with the $\lambda\mu\nu$ axes, the $\xi\eta\zeta$ set is rotated counterclockwise by an angle ϕ about the ζ axis. The second rotation is counterclockwise by an angle θ about the ξ axis; whereas, the third rotation is counterclockwise by an angle ψ about the ζ axis. Also, it should be noted that

[1]Figure 4-6 from *Classical Mechanics* by Herbert Goldstein, Copyright 1950 by Addison-Wesley Publishing Company, Inc., Reading, Mass. U.S.A. Permission to use this figure in this report only granted by the copyright owner, Addison-Wesley Publishing Company, Inc.

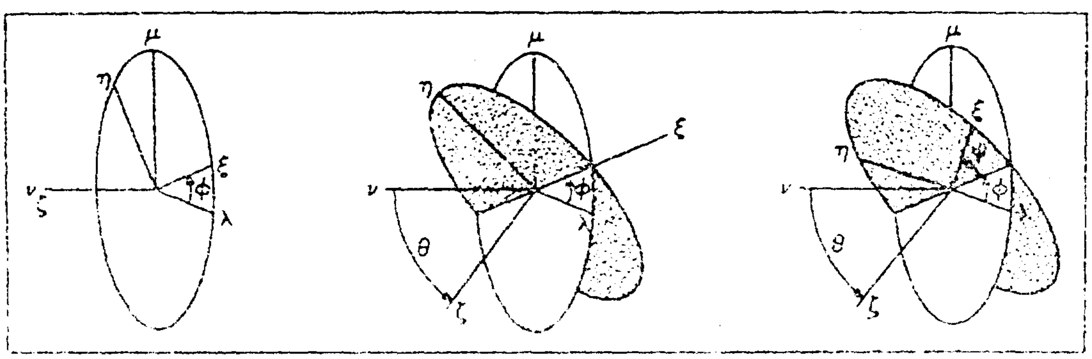

Fig. 3. Orientation Angles

the $\lambda\mu\nu$ coordinate system is obtained by rotating the xyz coordinate system counterclockwise by the angle γ about the z axis.

The matrix $A\ (\gamma,\ \phi,\ \theta,\ \psi)$ is used to transform vectors from xyz coordinates to $\lambda\mu\nu$ coordinates; whereas, its inverse $A^{-1}\ (\gamma,\ \phi,\ \theta,\ \psi)$ is used to perform the inverse transformation. These matrices are:

$$A = \begin{vmatrix} \cos\psi\ \cos(\phi+\gamma) - \cos\theta\ \sin(\phi+\gamma)\ \sin\psi & \cos\psi\ \sin(\phi+\gamma) + \cos\theta\ \cos(\phi+\gamma)\ \sin\psi & \sin\psi\ \sin\theta \\ -\sin\psi\ \cos(\phi+\gamma) - \cos\theta\ \sin(\phi+\gamma)\ \cos\psi & -\sin\psi\ \sin(\phi+\gamma) + \cos\theta\ \cos(\phi+\gamma)\ \cos\psi & \cos\psi\ \sin\theta \\ \sin\theta\ \sin(\phi+\gamma) & -\sin\theta\ \cos(\phi+\gamma) & \cos\theta \end{vmatrix}$$

$$A^{-1} = \begin{vmatrix} \cos\psi\ \cos(\phi+\gamma) - \cos\theta\ \sin(\phi+\gamma)\ \sin\psi & -\sin\psi\ \cos(\phi+\gamma) - \cos\theta\ \sin(\phi+\gamma)\ \cos\psi & \sin\theta\ \sin(\phi+\gamma) \\ \cos\psi\ \sin(\phi+\gamma) + \cos\theta\ \cos(\phi+\gamma)\ \sin\psi & -\sin\psi\ \sin(\phi+\gamma) + \cos\theta\ \cos(\phi+\gamma)\ \cos\psi & -\sin\theta\ \cos(\phi+\gamma) \\ \sin\psi\ \sin\theta & \cos\psi\ \sin\theta & \cos\theta \end{vmatrix}$$

II. TRAJECTORY MOTION

If motion in the z direction is small, the Equations describing the trajectory of the rocket center of mass can be written in the cylindrical coordinates as

$$\left.\begin{aligned} \frac{dy}{dt} &= \frac{v_x}{r} \\ \frac{dr}{dt} &= v_y \\ \frac{dz}{dt} &= v_z \end{aligned}\right\} \qquad (1)$$

$$\left.\begin{aligned} \frac{dv_x}{dt} &= \frac{D}{m} - \frac{v_x v_y}{r} \\ \frac{dv_y}{dt} &= \frac{L}{m} + \frac{v_x^2}{r} - g_0\left(\frac{r_0}{r}\right)^2 \\ \frac{dv_z}{dt} &= \frac{S}{m} \end{aligned}\right\} \qquad (2)$$

where

v_x = component of velocity vector along x axis

v_y = component of velocity vector along y axis

v_z = component of velocity vector along z axis

D = component of aerodynamic force vector along x axis

L = component of aerodynamic force along y axis

S = component of aerodynamic force along z axis

m = coasting mass of the re-entry body

g_0 = sea-level acceleration of gravity

r_0 = sea-level radius of the earth

III. ANGULAR MOTION

Euler's Equations are used to describe the angular motion of the body about its center of mass

$$\left.\begin{aligned} I_\xi \frac{d\omega_\xi}{dt} - \omega_\eta\, \omega_\zeta\, (I_\eta - I_\zeta) &= M_\xi \\ I_\eta \frac{d\omega_\eta}{dt} - \omega_\zeta\, \omega_\xi\, (I_\zeta - I_\xi) &= M_\eta \\ I_\zeta \frac{d\omega_\zeta}{dt} - \omega_\xi\, \omega_\eta\, (I_\xi - I_\eta) &= M_\zeta \end{aligned}\right\} \qquad (3)$$

where

I_i = principal moment of inertia about the ith axis

ω_i = component of angular velocity vector along the ith axis

M_i = moment along the ith axis

In order to obtain the Equations of motion in terms of Euler angles, new components for the angular velocity vector are defined.

$$\left.\begin{aligned} \frac{d\phi}{dt} &= \omega_\phi \\ \frac{d\theta}{dt} &= \omega_\theta \\ \frac{d\psi}{dt} &= \omega_\psi \end{aligned}\right\} \qquad (4)$$

In the body system of coordinates, ω_ϕ has the components

$$(\omega_\phi)_\xi = \omega_\phi \sin\theta \sin\psi \qquad (\omega_\phi)_\eta = \omega_\phi \sin\theta \cos\psi \qquad (\omega_\phi)_\zeta = \omega_\phi \cos\theta$$

ω_θ has t' e components

$$(\omega_\theta)_\xi = \omega_\theta \cos\psi \qquad (\omega_\theta)_\eta = -\omega_\theta \sin\psi \qquad (\omega_\theta)_\zeta = 0$$

and ω_ψ has the components

$$(\omega_\psi)_\xi = 0 \qquad (\omega_\psi)_\eta = 0 \qquad (\omega_\psi)_\zeta = \omega_\psi$$

Adding these components, the components of the angular velocity vector in body coordinates are as follows:

$$\left.\begin{aligned} \omega_\xi &= \omega_\phi \sin\theta \sin\psi + \omega_\theta \cos\psi \\ \omega_\eta &= \omega_\phi \sin\theta \cos\psi - \omega_\theta \sin\psi \\ \omega_\zeta &= \omega_\phi \cos\theta + \omega_\psi \end{aligned}\right\} \qquad (5)$$

After taking time derivations, it follows that

$$
\left.
\begin{aligned}
\sin\theta \frac{d\omega_\phi}{dt} &= \sin\psi \frac{d\omega_\xi}{dt} + \cos\psi \frac{d\omega_\eta}{dt} - \omega_\theta(\omega_\phi \cos\theta - \omega_\psi) \\
\frac{d\omega_\theta}{dt} &= \cos\psi \frac{d\omega_\xi}{dt} - \sin\psi \frac{d\omega_\eta}{dt} - \omega_\phi \, \omega_\psi \sin\theta \\
\frac{d\omega_\psi}{dt} &= \frac{d\omega_\zeta}{dt} - \frac{d}{dt}[\omega_\phi \cos\theta]
\end{aligned}
\right\} \qquad (6)
$$

Combining Eqs. (3) with Eqs. (6) gives

$$
\left.
\begin{aligned}
\sin\theta \frac{d\omega_\phi}{dt} &= \frac{\sin\psi}{I_\xi}[M_\xi + (I_\eta - I_\zeta)\,\omega_\eta\,\omega_\zeta] + \frac{\cos\psi}{I_\eta}[M_\eta + (I_\zeta - I_\xi)\,\omega_\zeta\,\omega_\xi] - \omega_\theta(\omega_\phi \cos\theta - \omega_\psi) \\
\frac{d\omega_\theta}{dt} &= \frac{\cos\psi}{I_\xi}[M_\xi + (I_\eta - I_\zeta)\,\omega_\eta\,\omega_\zeta] - \frac{\sin\psi}{I_\eta}[M_\eta + (I_\zeta - I_\xi)\,\omega_\zeta\,\omega_\xi] - \omega_\phi\,\omega_\psi \sin\theta \\
\frac{d\omega_\zeta}{dt} &= \frac{1}{I_\zeta}[M_\zeta + (I_\xi - I_\eta)\,\omega_\xi\,\omega_\eta] - \frac{d}{dt}[\omega_\phi \cos\theta]
\end{aligned}
\right\} \qquad (7)
$$

The rotational motion can be completely described by Eqs. (4), (5) and (7).

IV. AERODYNAMIC FORCES AND MOMENTS

Aerodynamic force and moment coefficients are usually available in body coordinates; therefore, it is necessary to find the components of the stream velocity vector in body coordinates

$$\begin{vmatrix} u_\xi \\ u_\eta \\ u_\zeta \end{vmatrix} = A \begin{vmatrix} -(v_x - w_x) \\ -v_y \\ -(v_z - w_z) \end{vmatrix} \tag{8}$$

$$\left.\begin{aligned}
u_\xi &= -(v_x - w_x)[\cos\psi \cos(\phi + \gamma) - \cos\theta \sin(\phi + \gamma) \sin\psi] \\
&\quad - v_y [\cos\psi \sin(\phi + \gamma) + \cos\theta \cos(\phi + \gamma) \sin\psi] - (v_z - w_z) \sin\psi \sin\theta \\
u_\eta &= \ \ (v_x - w_x)[\sin\psi \cos(\phi + \gamma) + \cos\theta \sin(\phi + \gamma) \cos\psi] \\
&\quad + v_y [\sin\psi \sin(\phi + \gamma) - \cos\theta \cos(\phi + \gamma) \cos\psi] - (v_z - w_z) \cos\psi \sin\theta \\
u_\zeta &= -(v_x - w_x) \sin\theta \sin(\phi + \gamma) + v_y \sin\theta \cos(\phi + \gamma) - (v_z - w_z) \cos\theta
\end{aligned}\right\} \tag{9}$$

In body coordinates, the aerodynamic forces are then found from

$$F_i = \frac{u_i}{|u_i|} \frac{1}{2} \rho(r) \left[u_\xi^2 + u_\eta^2 + u_\zeta^2\right] d^2 C_i(\alpha, \beta, \delta, M, Re) \qquad i = \xi, \eta, \zeta \tag{10}$$

where

α = angle of attack

β = roll angle

δ = fin deflection

Equations (2) require aerodynamic force components in space coordinates; therefore F_ξ, F_η, and F_ζ must be transformed to space coordinates

$$\begin{vmatrix} D \\ \\ L \\ \\ S \end{vmatrix} = A^{-1} \begin{vmatrix} F_\xi \\ \\ F_\eta \\ \\ F_\zeta \end{vmatrix} \tag{11}$$

Since the aerodynamic coefficients are functions of angle of attack and roll angle, it is necessary to determine these angles.

$$\left.\begin{aligned} D &= F_\xi \left[\cos\psi \cos(\phi + \gamma) - \cos\theta \sin(\phi + \gamma) \sin\psi\right] - F_\eta \\ &\quad \left[\sin\psi \cos(\phi + \gamma) + \cos\theta \sin(\phi + \gamma) \cos\psi\right] + F_\zeta \sin\theta \sin(\phi + \gamma) \\ L &= F_\xi \left[\cos\psi \sin(\phi + \gamma) + \cos\theta \cos(\phi + \gamma) \sin\psi\right] - F_\eta \\ &\quad \left[\sin\psi \sin(\phi + \gamma) - \cos\theta \cos(\phi + \gamma) \cos\psi\right] - F_\zeta \sin\theta \cos(\phi + \gamma) \\ S &= F_\xi \sin\theta \sin\psi + F_\eta \sin\theta \cos\psi + F_\zeta \cos\theta \end{aligned}\right\} \tag{12}$$

$$\cos\alpha = \frac{-u_\xi}{\left(u_\xi^2 + u_\eta^2 + u_\zeta^2\right)^{1/2}} \qquad \cos\beta = \frac{-u_\eta}{\left(u_\eta^2 + u_\zeta^2\right)^{1/2}} \tag{13}$$

The pitching moment is given by

$$M_\zeta = \frac{-u_\eta}{\left|u_\eta\right|} \frac{1}{2} \rho\left(u_\xi^2 + u_\eta^2 + u_\zeta^2\right) d^3 C_{M\zeta} (\alpha, \beta, \delta, M, Re) \tag{14}$$

the yawing moment by

$$M_\eta = \frac{u_\zeta}{\left|u_\zeta\right|} \frac{1}{2} \rho\left(u_\xi^2 + u_\eta^2 + u_\zeta^2\right) d^3 C_{M\eta} (\alpha, \beta, \delta, M, Re) \tag{15}$$

and the rolling moment by

$$M_\xi = \frac{1}{2} \rho \left(u_\xi^2 + u_\eta^2 + u_\zeta^2\right) d^2 \, l \, C_{M\xi} \, (\alpha, \beta, \delta, M, Re) \tag{16}$$

V. SUMMARY

The trajectory Equations (1 and 2) simultaneously with the rotation Equations (4, 5, and 7) describe the dynamics of the coasting rocket. The aerodynamic forces are given by Eqs. (10) and (12), whereas, the aerodynamic moments are obtained from Eqs. (14), (15), and (16). The forces and moments are functions of angle of attack, roll angle, and fin deflection. Angle of attack and roll angle are computed by Eq. (13); whereas, fin deflection is a programmed input. All aerodynamic coefficients must be supplied as inputs.

NOMENCLATURE

$A(\gamma, \phi \quad, \psi)$ = transformation matrix from xyz axes $\xi\eta\zeta$ axes.

$A^{-1}(\gamma, \phi, \theta, \psi)$ = transformation matrix from $\xi\eta\zeta$ axes to xyz axes.

C_ζ = aerodynamic coefficient of normal force in yaw plane.

C_η = aerodynamic coefficient of normal force in pitch plane.

C_ξ = aerodynamic coefficient of cord force.

$C_{M\zeta}$ = aerodynamic coefficient of pitch about center of mass.

$C_{M\eta}$ = aerodynamic coefficient of yaw about center of mass.

$C_{M\xi}$ = aerodynamic coefficient of roll about center of mass.

d = rocket base diameter.

D = aerodynamic force component along x axis.

F_ζ = aerodynamic normal force in yaw plane.

F_η = aerodynamic normal force in pitch plane.

F_ξ = aerodynamic cord force.

g_0 = sea-level acceleration of gravity.

I_ζ = pitch moment of inertia.

I_η = yaw moment of inertia.

I_ξ = roll moment of inertia.

l = distance between missile axis and fin center of pressure.

L = aerodynamic force component along y axis.

m = mass of coasting rocket.

M = Mach number.

M_ζ = pitch moment, see Eq. (14).

M_η = yaw moment, see Eq. (15).

M_ξ = roll moment, see Eq. (16).

r_0 = sea-level radius of earth.

NOMENCLATURE (Cont'd)

Re = Reynolds number.

r, y, z = earth-fixed secondary coordinate system (Fig. 1).

S = aerodynamic force component along z axis.

u_ξ, u_η, u_ζ = components of stream velocity in body coordinates.

v_x = center of mass velocity vector component along x axis.

v_y = center of mass velocity vector component along y axis.

v_z = center of mass velocity vector component along z axis.

w_x = downrange wind velocity.

w_z = crossrange wind velocity.

x, y, z = local space coordinates associated with r, y, z (Fig. 1).

X, Y, Z = earth-fixed primary coordinate system (Fig. 1).

α = angle of attack: angle between stream velocity vector and ξ axis.

β = roll angle: angle between plane of angle of attack and η axis.

δ = fin deflection.

λ, μ, ν = local space coordinates associated with X, Y, Z (Figs. 1 and 3).

ξ, η, ζ = body-fixed coordinate system (Fig. 2).

$\rho(r)$ = atmospheric density.

ϕ, θ, ψ = Euler orientation angles (Fig. 3).

ω_ζ = pitch rate.

ω_η = yaw rate.

ω_ξ = roll rate.

$\omega_\phi, \omega_\theta, \omega_\psi$ = see Eq. (4).